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THE HEAD WAVE AT THE BOUNDARY OF TWO HEREDITARY-ELASTIC HALF-SPACES. THE CASE OF A LINEAR SOURCE†

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The asymptotic form at the front for reflected and head waves is constructed in the problem of the reflection of a cylindrical, horizontally polarized, shear wave (a so-called *SH*-wave) from the interface between two homogeneous isotropic hereditary-elastic half spaces which are joined to one another and for which the heredity kernels are assumed to be regular. This problem is solved here by the generalized Cagniard–de Hoop (CH) method which is an improved form of the approach proposed in [1]. Unlike [1], in the generalization of the CH method proposed here, it is not necessary for the memory functions to be the same in all of the layers of a laminar medium.

The solution of this problem was obtained in [2] in the purely elastic case using the classical CH method. It should be noted that the results in this paper could also have been obtained by another method which rest on the results in [3] where the high-frequency asymptotic form of the potential of the reflected and refracted waves was constructed for the case of a harmonic source. However, straightforward derivation of the asymptotic forms at the front based on the generalization of the CH method presented in this note turns out to be simpler.

1. In xyz space (with the Z-axis directed vertically downwards), consider two homogeneous isotropic hereditary-elastic half-spaces which are rigidly joined to one another along the surface $z=0$. We recall that, in a homogeneous isotropic hereditary-elastic medium, the operational shear modulus μ^* defines (in the case of pure shear) the following relation between the stresses and strains

$$\sigma_{ik}=2\mu^*\varepsilon_{ik}=2\mu^0(1-Q(t)*)\varepsilon_{ik}$$

Here $\mu^0=\text{const}$ is the instantaneous shear modulus, $Q(t)$ is the shear relaxation kernel, which is equal to zero when $t>0$, and an asterisk in a row denotes convolution with respect to t (the notation of [1] is used).

We shall therefore assume that the half-space $z<0$ possesses a density ρ_1 and a shear modulus $\mu_1^*=\mu_2^0(1-Q_1(t)^*)$ while the half-space possesses a density ρ_2 and a shear modulus $\mu_2^*=\mu_2^0(1-Q_2(t)^*)$. We assume that the relaxation kernels $Q_j(t)\geqslant 0$ are regular, that is, they possess finite limits when $t\to+0$. At the same time we assume that, when $t>0$, the expansions

$$Q_j(t)=Q_j(0)+\sum_{k\geqslant 1}\kappa_{jk}t^k$$

are valid (here and subsequently, we shall write $Q_j(0)$ instead of $Q_j(+0)$, for brevity).

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Suppose a linear source

$$f=(0,A\delta(x)\delta(z-z_0)\delta(t),0);\quad z_0<0$$

acts in the upper half-space.

Such a source only excites the y-component of the displacement (which we shall denote by υ). It is clear that $\upsilon=(x, z, t)$ in the problem being considered. The equations for υ in the corresponding half-spaces have the form

$$\begin{aligned}\rho_1\partial^2\upsilon/\partial t^2 &= A\delta(x)\delta(z-z_0)\delta(t)+\mu_1^*\Delta\upsilon,\quad z<0\\ \rho_2\partial^2\upsilon/\partial t^2 &= \mu_2^*\Delta\upsilon,\quad z>0\end{aligned}\tag{1.1}$$

where Δ is the Laplace operator in x and z. Here, we assume that

$$\upsilon=0 \text{ when } t<0 \tag{1.2}$$

Next, the stress σ_{yz} is related to the displacement υ by the formulae

$$\sigma_{yz}=\mu_1^*\partial\upsilon/\partial z,\quad z<0;\quad \sigma_{yz}=\mu_2^*\partial\upsilon/\partial z,\quad z>0$$

The condition of continuity of the stresses on the boundary $z=0$ therefore takes the form

$$\mu_1^*\partial\upsilon/\partial z|_{z=0-}=\mu_2^*\partial\upsilon/\partial z|_{z=0+} \tag{1.3}$$

Finally, the condition of continuity of the displacements has the form

$$\upsilon|_{z=0-}=\upsilon|_{z=0+} \tag{1.4}$$

2. Let us apply a double Fourier–Laplace transformation $x,\ t\to\xi,\ s$ to problem (1.1)–(1.4)

$$\int_{-\infty}^{\infty} e^{-i\xi x}dx\int_0^{\infty} e^{-st}dt$$

As a result, problem (1.1)–(1.4) reduces to matching the solutions of ordinary differential equations with respect to z. It can be verified that the solution of this problem has the form

$$\upsilon(\xi,z,s)=\begin{cases}\dfrac{A}{2\mu_1(s)n_1}\exp[-n_1|z-z_0|]+\upsilon_{\text{ref}}(\xi,z,s), & z<0\\ \upsilon_{\text{tr}}(\xi,z,s), & z>0\end{cases}\tag{2.1}$$

$$\upsilon_{\text{ref}}(\xi,z,s)=\frac{k_{\text{ref}}A}{2\mu_1(s)n_1}\exp[n_1(z+z_0)],\quad \upsilon_{\text{tr}}(\xi,z,s)=\frac{k_{\text{tr}}A}{2\mu_1(s)n_1}\exp[-(n_2z+n_1z_0)]$$

Here

$$\begin{aligned}&n_j=(\xi^2+s^2/\beta_j^2(s))^{1/2},\quad \operatorname{Re}n_j\geqslant 0\\ &\beta_j(s)=(\mu_j(s)/\rho_j)^{1/2},\quad \mu_j(s)=\mu_j^0(1-Q_j(s));\quad j=1,2\\ &k_{\text{ref}}=\frac{\mu_1(s)n_1-\mu_2(s)n_2}{\mu_1(s)n_1+\mu_2(s)n_2},\quad k_{\text{tr}}=\frac{2\mu_1(s)n_1}{\mu_1(s)n_1+\mu_2(s)n_2}\end{aligned}\tag{2.2}$$

(k_{ref} and k_{tr} are the reflection and transmission coefficients).

Below, notation is also needed for the instantaneous elastic velocities of the shear waves $\beta_j^0 = (\mu_j^0/\rho_j^0)^{1/2}$. It is clear that

$$\beta_j(s) = \beta_j^0(1 - Q_j(s))^{1/2} \tag{2.3}$$

Henceforth, we shall consider the case when

$$\beta_2^0 > \beta_1^0 \tag{2.4}$$

which, in the problem in question, corresponds to the occurrence of head waves. It obviously follows from (2.4) that

$$\beta_2(s) > \beta_1(s) \quad \text{when } s \to +\infty \tag{2.5}$$

3. We will confine ourselves to investigating the wave reflected from the boundary in the half-space $z < 0$, that is, to investigating the function $\upsilon_{ref}(\xi, z, s)$ defined in (2.1). We apply an inverse Fourier transformation with respect to x to the expression for $\upsilon_{ref}(\xi, z, s)$ and then introduce a radial parameter p using the same formula as in the purely elastic case [2]; $\xi = isp$. Now, using the notation

$$\eta_j = (\beta_j^{-2}(s) - p^2)^{1/2}, \quad \operatorname{Re}\eta_j \geqslant 0$$

we note that (for purely imaginary p) the real part of the integrand for $\upsilon_{ref}(x, z, s)$ is an even function of p, while the imaginary part is an odd function of p. We therefore obtain

$$\upsilon_{ref}(x,z,s) = \operatorname{Im}\int_0^{i\infty} \frac{\Phi(\mu_j(s),\eta_j)}{\eta_1}\exp[-s(px + \eta_1|z + z_0|)]dp \tag{3.1}$$

$$\Phi(\alpha_j, \beta_j) = \frac{A}{2\pi\alpha_1}\,\frac{\alpha_1\beta_1 - \alpha_2\beta_2}{\alpha_1\beta_1 + \alpha_2\beta_2}$$

We also note that, henceforth, during the deformation of the integration contour in (3.1), it is necessary to take account of the cuts

$$1/\beta_1(s) \leqslant p < \infty, \quad 1/\beta_2(s) \leqslant p < \infty \tag{3.2}$$

which are made along the $\operatorname{Re} p$ axis and are connected with the requirement that $\operatorname{Re}\eta_1 \geqslant 0$, $\operatorname{Re}\eta_2 \geqslant 0$.

4. We will introduce the real variable τ, $0 < \tau < \infty$ and, for each $s = \text{const} > 0$, we consider the equation

$$\tau = px + \eta_1|z + z_0| \tag{4.1}$$

It can be shown that one of the solutions of Eq. (4.1) for p has the form

$$p_s = R_0^{-2}[x\tau + \kappa|z + z_0|\,|R_0^2/\beta_1^2(s) - \tau^2|^{1/2}]$$

$$\kappa = \begin{cases} -1, & 0 < \tau \leqslant R_0/\beta_1(s) \\ i, & \tau > R_0/\beta_1(s) \end{cases} \tag{4.2}$$

$$R_0 = [x^2 + (z + z_0)^2]^{1/2}$$

The quantities x and z are considered here as fixed parameters and the parameter $s > 0$ is also fixed.

We shall refer to the contour $p = p_s(\tau)$, defined by formulae (4.2), as the generalized Cagniard contour for υ_{ref}. The variable τ plays the same role in (4.2) as the time t in the corresponding formulae of the classical CH method for purely elastic media. Note that, in (4.2), the "similar to time" variable τ and the Laplace variable s occur simultaneously.

Let us denote the part of the generalized Cagniard contour (4.2) lying in the half plane $\operatorname{Re} p \geqslant 0$ by C_s. As in the classical CH method [2], we conclude that formula (3.1) can be rewritten in the form

$$\upsilon_{ref}(x,z,s) = \operatorname{Im} \int_{C_s} \frac{\Phi(\mu_j(s),\eta_j)}{\eta_1} dp_s(\tau) \tag{4.3}$$

5. To fix our ideas, let $x > 0$. It is then obvious from (4.2) that the contour C_s departs from the Re p axis at the point

$$\bar{p}_s = x / (R_0 \beta_1(s)) > 0 \tag{5.1}$$

Let us assume that

$$x / R_0 < \beta_1^0 / \beta_2^0 \tag{5.2}$$

Then, for large $s > 0$, the point (5.1) lies to the left of both the cuts (3.2) which have been made in the half plane Re $p > 0$, that is

$$x / (R_0 \beta_1(s)) < 1/\beta_2(s) < 1/\beta_1(s), \quad s \to +\infty$$

As is shown below, satisfaction of the inequality (5.2) means that x is smaller than the critical distance at which a head wave is observed.

If condition (5.2) is satisfied then, for large $s > 0$ in the segment C_s lying on the Re p axis, the integrand in (4.3) is real and, consequently, makes no contribution to the expression for υ_{ref}. Hence, for large $s > 0$, it may be assumed that $\tau > R_0/\beta_1(s)$ in (4.3). Next, we have

$$dp_s / d\tau = i\eta_1 [\tau^2 - R_0^2 / \beta_1^2(s)]^{-1/2} \quad \text{for} \quad \tau > R_0 / \beta_1(s) \tag{5.3}$$

Hence, for large $s > 0$, (4.3) acquires the form

$$\upsilon_{ref}(x,z,s) = \int_{R_0/\beta_1(s)}^{\infty} \operatorname{Re}\Phi(\mu_j(s),\eta_j) \frac{e^{-s\tau} d\tau}{(\tau^2 - R_0^2/\beta_1^2(s))^{1/2}} \tag{5.4}$$

6. We will now calculate the asymptotic form at the front $\upsilon_{ref}(x, z, t)$, assuming that condition (5.2) is satisfied. To do this, we make the following substitution in (5.4)

$$\tau \to \tau \beta_1^0 / \beta_1(s) \tag{6.1}$$

By virtue of (2.3)

$$\exp[-s\tau\beta_1^0 / \beta_1(s)] = e^{-s\tau} \exp[-\tau Q_1(0)/2](1 + O(s^{-1}))$$

Now, from (5.4), we obtain

$$\upsilon_{ref}(x,z,s) \sim \int_{R_0/\beta_1^0}^{\infty} \operatorname{Re}\Phi(\mu_j^0,\eta_j^0(\tau)) \frac{\exp[-\tau Q_1(0)/2]}{[\tau^2 - R_0^2/(\beta_1^0)^2]^{1/2}} e^{-s\tau} d\tau$$

$$s \to +\infty \qquad (6.2)$$

$$\eta_j^0(\tau) = [(\beta_j^0)^{-2} - p_\infty^2(\tau)]^{1/2}$$

Guided by the uniqueness theorem for the representation of a Laplace transform, we conclude that, close to the front of the reflected wave (that is, when $t \to R_0/\beta_1^0 + 0$)

$$\upsilon_{\text{ref}}(x,z,t) \sim \operatorname{Re}\Phi(\mu_j^0, \eta_j^0(t)) \frac{\exp[-Q_1(0)R_0/(2\beta_1^0)]}{[t^2 - R_0^2/(\beta_1^0)^2]^{1/2}} \qquad (6.3)$$

It is seen that the exponential factor, determining the decay at the front, does not contain the quantity $Q_2(0)$. Expression (6.3) therefore corresponds to a wave which only propagates in the half space $z < 0$.

7. Now let

$$x/R_0 > \beta_1^0/\beta_2^0 \qquad (7.1)$$

Then, the point $\overline{p}_s$, at which the contour C_s leaves the $\operatorname{Re} p$ axis (see (5.1)) satisfies the inequalities

$$1/\beta_2(s) < \overline{p}_s = x/R_0\beta_1(s) < 1/\beta_1(s), \quad s \to +\infty \qquad (7.2)$$

that is, for large $s > 0$, it lies between the ends of the cuts (3.2) located in the half plane $\operatorname{Re} p > 0$. Then, for large $s > 0$, and in the rectilinear segment $[1/\beta_2(s), \overline{p}_s]$ of the contour C_s, the quantity $\eta_2 = [\beta_2^{-2}(s) - p^2]^{1/2}$ turns out to be purely imaginary. Unlike (5.3), for large $s > 0$ in the indicated segment of the contour C_s, we have

$$dp_s/d\tau = \eta_1[R_0^2/\beta_1^2(s) - \tau^2]^{-1/2} \qquad (7.3)$$

Now, substituting (5.3) and (7.3) into (4.3), we obtain, for large $s > 0$

$$\upsilon_{\text{ref}}(x,z,s) = \operatorname{Im} \int_{\tau_s}^{\infty} \Phi(\mu_j(s), \eta_j) \frac{e^{-s\tau}}{[R_0^2/\beta_1^2(s) - \tau^2]^{1/2}} \qquad (7.4)$$

Here, the quantity τ_s is determined from (4.1) where p has to be replaced by $1/\beta_2(s)$

$$\tau_s = x/\beta_2(s) + [\beta_1^{-2}(s) - \beta_2^{-2}(s)]^{1/2}|z + z_0| \qquad (7.5)$$

Furthermore, when $\tau > R_0/\beta_1(s)$ by definition we assume that

$$[R_0^2/\beta_1^2(s) - \tau^2]^{1/2} = i[\tau^2 - R_0^2/\beta_1^2(s)]^{1/2} \qquad (7.6)$$

We now introduce the notation

$$\tau_0 = x/\beta_2^0 + [(\beta_1^0)^{-2} - (\beta_2^0)^{-2}]^{1/2}|z + z_0|$$

(As will become clear below, $t = \tau_0$ is the time of arrival of the head wave at a point (x, y, z).). Making the substitution $\tau \to \tau\tau_s/\tau_0$ when $s \to +\infty$ we transform relation (7.4) into

$$\upsilon_{\text{ref}}(x,z,s) \sim \int_{\tau_0}^{\infty} \operatorname{Im}\{\Phi(\mu_j^0, \eta_j^0(\tau))[R_0^2/(\beta_1^0)^2 - \tau^2]^{-1/2}\}\exp(-s\tau\tau_s/\tau_0)d\tau \qquad (7.7)$$

Now, by taking account of the fact that, for large $s > 0$

$$\beta_2^0\tau_s = \beta_2^0\tau_0 + (2s)^{-1}Q_2(0)\{x - |z + z_0|[(\beta_2^0/\beta_1^0)^2 - 1]^{-1/2}\} +$$
$$+(2s)^{-1}Q_1(0)(\beta_2^0/\beta_1^0)^2[(\beta_2^0/\beta_1^0)^2 - 1]^{-1/2} + O(s^{-2})$$

we obtain from (7.7) the required frontal asymptotic form of the head wave $(t \to \tau_0 + 0)$

$$\begin{aligned} \upsilon_{ref}(x,z,t) &\sim \mathrm{Im}\{\Phi(\mu_j^0, \eta_j^0(t))[R_0^2/(\beta_1^0)^2 - t^2]^{-1/2}\}\times \\ &\times\exp\{-(2\beta_2^0)^{-1}Q_2(0)[x - |z + z_0|((\beta_2^0/\beta_1^0)^2 - 1)^{-1/2}] - \\ &-(2\beta_1^0)^{-1}Q_1(0)|z + z_0|(1 - (\beta_1^0/\beta_2^0)^2)^{-1/2}\} \end{aligned} \tag{7.8}$$

Here, we assume that $x > 0$ and, moreover, that relationship (7.6) is satisfied. It is obvious that (7.8), unlike (6.3), is a wave which has passed through both media. It is clear that the distance traversed by this wave through the lower medium is equal to $x - |z + z_0|((\beta_2^0/\beta_1^0)^2 - 1)^{-1/2}$.

8. Finally, the case when $\beta_1^0 = \beta_2^0$ but $\beta_1(s) < \beta_2(s)$ (for example, in a ray $s_0 < s < \infty$) is interesting. To be specific, let $x > 0$. It is clear that $x/R_0 < 1$ everywhere. The case (5.2) is thereby realized, while the case (7.1) is impossible, that is, there is no head wave in the problem in question.

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